

Spectral properties of non-selfadjoint extensions of the Calogero Hamiltonian

Giorgio Metafune* and Motohiro Sobajima†

Abstract. We describe all extensions of the Calogero Hamiltonian

$$L = -\frac{d^2}{dr^2} + \frac{b}{r^2} \quad \text{in } L^2(\mathbb{R}_+), \quad b < -\frac{1}{4}$$

having non empty resolvent and generating an analytic semigroup in $L^2(\mathbb{R}_+)$.

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1 Introduction

We study spectral properties of the Calogero Hamiltonian in $\mathbb{R}_+ := (0, \infty)$, that is of one-dimensional Schrödinger operator with inverse square potentials

$$L = -\frac{d^2}{dr^2} + \frac{b}{r^2} \quad \text{in } L^2(\mathbb{R}_+),$$

where $b \in \mathbb{R}$. By Hardy's inequality the quadratic form

$$\int_0^\infty \left(|u'(r)|^2 + \frac{b}{r^2} |u(r)|^2 \right) dr$$

is nonnegative on $D(L_{\min}) := C_0^\infty(\mathbb{R}_+)$ if and only if $b \geq -\frac{1}{4}$. In this case the Friedrichs extension of L_{\min} is selfadjoint and nonnegative. Moreover, if $b \geq \frac{3}{4}$, then L_{\min} is essentially selfadjoint. In N -dimensional case, the threshold for nonnegativity of

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + \frac{b}{|x|^2} |u(x)|^2 \right) dx, \quad u \in D(L_{\min}) := C_c^\infty(\mathbb{R}^N \setminus \{0\})$$

is $-(\frac{N-2}{2})^2$ and that for the essential selfadjointness of $L_{\min} = -\Delta + b|x|^{-2}$ is $-(\frac{N-2}{2})^2 + 1$. These constants are the optimal constants of Hardy's and Rellich's inequalities, respectively, see [12]. On the other hand if $b < -(\frac{N-2}{2})^2$, Baras and Goldstein proved in [2] that there is no positive distributional solution of the equation

$$u_t(x, t) - \Delta u(x, t) + \frac{b}{|x|^2} u(x, t) = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}_+ \quad (1.1)$$

*Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Via Per Arnesano, 73100, Lecce, Italy, E-mail: giorgio.metafune@unisalento.it

† Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Via Per Arnesano, 73100, Lecce, Italy, E-mail: msobajima1984@gmail.com

apart from the zero solution. This nonexistence result for positive solutions has been generalized by subsequent papers ([4], [7], [8], [9] and [6]). Since for $b < -\left(\frac{N-2}{2}\right)^2$ the quadratic form above is unbounded from below, every selfadjoint extension of L_{\min} has a spectrum unbounded from below and cannot be the (minus) generator of a semigroup.

In this paper we mainly consider the one dimensional case and assume that

$$b < -\frac{1}{4} \quad \nu := \sqrt{-\frac{1}{4} - b} > 0. \quad (1.2)$$

We characterize all intermediate operators between L_{\min} and $L_{\max} := (L_{\min})^*$, given by

$$D(L_{\max}) := \{u \in L^2(\mathbb{R}_+) \cap H_{\text{loc}}^2(\mathbb{R}_+) ; Lu \in L^2(\mathbb{R}_+)\},$$

with non-empty resolvent set, including all selfadjoint extensions, and describe their spectrum. Spectral properties of selfadjoint extensions are also considered in [5] when $b < -\frac{1}{4}$. We show that there exist infinitely many non-selfadjoint extensions $-\tilde{L}$ which are generators of analytic semigroups. Since Hardy's inequality fails, these semigroups cannot be (quasi) contractive. Some partial results in the N -dimensional case are stated in the last section. Vazquez and Zuazua pointed out in [14] that the existence of solutions of (1.1) might require a lower bound of b and a restriction of initial data. Our result, in contrast, are valid for any $b \in \mathbb{R}$ and any initial datum in $L^2(\mathbb{R}^N)$.

2 Preliminaries

In this section we study the equation $\lambda u + Lu = f$.

2.1 The homogeneous equation

If $\lambda \notin]-\infty, 0]$ the above equation with $f = 0$ has two solutions, one exponential decaying, the other exponential growing at ∞ . The behavior of these two solutions near 0 is studied in the next two lemmas. To state them, for $\nu > 0$ we define

$$\alpha = \alpha(\nu) = c \frac{2^{-i\nu}}{\nu \Gamma(i\nu)} = c \frac{2^{-i\nu} i}{\Gamma(1 + i\nu)}, \quad (2.1)$$

where $c > 0$ is independent of ν and will play no role in what follows.

Lemma 2.1. *Let $\omega \in \mathbb{C}_+$, $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \pi/2$ and assume that (1.2) holds. Then there exists a solution $\varphi_{\omega,0}$ of*

$$\omega^2 \varphi(r) - \varphi''(r) + \frac{b}{r^2} \varphi(r) = 0, \quad r \in \mathbb{R}_+ \quad (2.2)$$

and a constant $R = R(b, \omega) > 0$ such that

$$|\varphi_{\omega,0}(r)| \leq 2e^{-(\text{Re } \omega)r}, \quad r \geq R. \quad (2.3)$$

Moreover $\varphi_{\omega,0}(r)$ is real when ω is real and

$$\left| r^{-\frac{1}{2}} \varphi_{\omega,0}(r) - \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}} \left(\alpha \mu^{i\nu} e^{-\xi\nu} r^{i\nu} + \bar{\alpha} \mu^{-i\nu} e^{\xi\nu} r^{-i\nu} \right) \right| \rightarrow 0 \quad \text{as } r \downarrow 0, \quad (2.4)$$

where α is defined in (2.1).

Proof. (Step 1). We consider the modified Bessel equation

$$w(z) - \frac{d^2 w}{dz^2}(z) - \frac{b}{z^2} w(z) = 0, \quad z \in \mathbb{C}_+. \quad (2.5)$$

The indicial equation $\alpha(\alpha - 1) = b$ has roots $\alpha_1 = \frac{1}{2} + i\sqrt{\nu}$ and $\alpha_2 = \frac{1}{2} - i\sqrt{\nu}$. Then every solution has the form

$$w(z) = g_1(z) z^{\frac{1}{2} + i\nu} + g_2(z) z^{\frac{1}{2} - i\nu}, \quad (2.6)$$

with g_1, g_2 entire functions, $g_1(0) \neq 0, g_2(0) \neq 0$, and therefore is holomorphic in $\mathbb{C} \setminus]-\infty, 0]$, see [3, Chapter 9.6, 9.8].

Let us show that there exists a solution of (2.5) which behaves like e^{-z} in $E_R := \{z \in \mathbb{C}_+ ; |z| > R\}$. Setting $h(z) := e^z w(z)$ (2.5) reduces to

$$\frac{d^2 h}{dz^2}(z) - 2 \frac{dh}{dz}(z) = \frac{b}{z^2} h(z), \quad z \in \mathbb{C}_+. \quad (2.7)$$

We indicate with $X := H^\infty(E_R)$, the set of all bounded holomorphic functions in E_R , endowed with $\|h\|_X := \sup_{z \in E_R} |h(z)|$. Define

$$Th(z) := 1 + \int_{\Gamma_z} e^{2\xi} \left(\int_{\Gamma_\xi} \frac{be^{-2\eta}}{\eta^2} h(\eta) d\eta \right) d\xi, \quad z \in E_R, \quad (2.8)$$

where $\Gamma_z := \{tz ; t \in [1, \infty)\}$; note that a fixed point of T satisfies (2.7). Then $T : X \rightarrow X$ is well-defined and contractive in X when R is large enough. In fact, if $h \in X$, then Th is well-defined and holomorphic in E_R . Moreover, for $z \in E_R$,

$$\begin{aligned} |Th(z) - 1| &= \left| \int_1^\infty e^{2tz} \left(\int_t^\infty \frac{be^{-2sz}}{(sz)^2} h(sz) z ds \right) z dt \right| = \left| \int_1^\infty \left(\int_1^s e^{2tz} dt \right) \frac{be^{-2sz}}{s^2} h(sz) ds \right| \\ &\leq \left| \frac{b(1 - e^{2(s-1)z})}{2z} \right| \left(\int_1^\infty \frac{1}{s^2} ds \right) \|h\|_X \leq \frac{|b|}{R} \|h\|_X. \end{aligned}$$

Similarly, we have $|Th_1(z) - Th_2(z)| \leq (|b|/R) \|h_1 - h_2\|_X$ for every $h_1, h_2 \in X$ and $z \in E_R$. Therefore $T : X \rightarrow X$ is well-defined and if we choose $R_0 := 2|b|$, then T is contractive. Let $h_0 \in X$ be the unique fixed point of T . Noting that

$$|h_0(z) - 1| = |Th_0(z) - T0(z)| \leq \frac{|b|}{R_0} \|h_0\|_X \leq \frac{\|h_0 - 1\|_X + 1}{2},$$

we deduce $\|h_0 - 1\|_X \leq 1$. Taking $w_0(z) := e^{-z} h_0(z)$ it follows that w_0 can be continued as a solution of (2.5) and

$$|e^z w_0(z)| \leq 2, \quad z \in E_{R_0}. \quad (2.9)$$

Now we define

$$\varphi_{\omega,0}(r) := w_0(\omega r), \quad r \in \mathbb{R}_+.$$

Then $\varphi_{\omega,0}$ solves (2.2)

$$\omega^2 \varphi_{\omega,0}(r) - \varphi_{\omega,0}''(r) + \frac{b}{r^2} \varphi_{\omega,0}(r) = \omega^2 \left(w_0(\omega r) - \frac{d^2 w_0}{dz^2}(\omega r) + \frac{b}{(\omega r)^2} w_0(\omega r) \right) = 0.$$

Moreover, if $r > R := R_0/|\omega|$, then

$$|e^{\omega r} \varphi_{\omega,0}(r)| = |e^{\omega r} w_0(\omega r)| \leq 2$$

and (2.3) is satisfied.

(Step 2). Next we consider w_0 on the positive real axis and we may assume that w_0 is real on it (otherwise we consider $\frac{1}{2}(w_0(z) + \overline{w_0(\bar{z})})$). By (2.6) we have

$$w_0(z) = g_1(z)z^{\frac{1}{2}+i\nu} + g_2(z)z^{\frac{1}{2}-i\nu}, \quad z \in \mathbb{C} \setminus]-\infty, 0] \quad (2.10)$$

where g_1, g_2 are entire functions. Then $g_1(r) = \overline{g_2(r)}$ for $r > 0$ and $\alpha = g_1(0) = \overline{g_2(0)} \neq 0$. This implies that

$$\left| z^{-\frac{1}{2}} w_0(z) - (\alpha z^{i\nu} + \bar{\alpha} z^{-i\nu}) \right| \rightarrow 0 \quad \text{as } z \rightarrow 0 \quad (z \in \mathbb{C}_+).$$

Consequently we obtain (2.4) with $K_{\omega,0} = \omega^{\frac{1}{2}} = \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}}$

$$\begin{aligned} & \left| r^{-\frac{1}{2}} \varphi_{\omega,0}(r) - \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}} \left(\alpha e^{-\xi\nu} \mu^{i\nu} r^{i\nu} + \bar{\alpha} e^{\xi\nu} \mu^{-i\nu} r^{-i\nu} \right) \right| \\ &= \mu^{\frac{1}{2}} \left| (\omega r)^{-\frac{1}{2}} w_0(\omega r) - (\alpha (\omega r)^{i\nu} + \bar{\alpha} (\omega r)^{-i\nu}) \right| \rightarrow 0 \quad \text{as } r \downarrow 0. \end{aligned}$$

(Step 3). Finally we show that α is given by (2.1). In fact $\varphi_{1,0}(r)$, being the unique (up to constants) exponentially decaying solution of (2.2) with $\omega = 1$, coincides with $cr^{\frac{1}{2}} K_{i\nu}(r)$, where $c > 0$ and $K_{i\nu}$ is the modified Bessel function of second kind. Therefore by [1, 9.6.2 and 9.6.7 in p. 375] we deduce that

$$r^{-\frac{1}{2}} \varphi_{1,0}(r) = \frac{c\pi(I_{-i\nu}(r) - I_{i\nu}(r))}{2\sin(i\nu\pi)} \sim c' \left(\frac{2^{-i\nu}}{\nu\Gamma(i\nu)} r^{i\nu} + \frac{2^{i\nu}}{\nu\Gamma(-i\nu)} r^{-i\nu} \right)$$

as $r \downarrow 0$ for some $c' > 0$. Therefore α is given by (2.1). \square

Next we investigate the behavior at 0 of the exponentially growing solution.

Lemma 2.2. *Let $\omega \in \mathbb{C}_+$ satisfy $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \pi/2$ and assume that (1.2) holds. Then there exist a solution $\varphi_{\omega,1}$ of (2.2) and constants $C'_\omega > C_\omega > 0$ and $R' > 0$ such that*

$$C_\omega e^{(\operatorname{Re} \omega)r} \leq |\varphi_{\omega,1}(r)| \leq C'_\omega e^{(\operatorname{Re} \omega)r} \quad \text{as } r \geq R', \quad (2.11)$$

$$\left| r^{-\frac{1}{2}} \varphi_{\omega,1}(r) - \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}} \left(\alpha \mu^{i\nu} e^{-\xi\nu} r^{i\nu} - \bar{\alpha} \mu^{-i\nu} e^{\xi\nu} r^{-i\nu} \right) \right| \rightarrow 0 \quad \text{as } r \downarrow 0, \quad (2.12)$$

where α is defined in (2.1). Finally, $i\varphi_{\omega,1}(r)$ is real when ω is real.

Proof. By (2.6) there exist two solutions w_1, w_2 satisfying

$$z^{-\frac{1}{2}-i\nu} w_1(z) \rightarrow 1, \quad z^{-\frac{1}{2}+i\nu} w_2(z) \rightarrow 1 \quad \text{as } z \rightarrow 0.$$

With the notation of the proof of Lemma 2.1 we have $\varphi_{\omega,0}(r) = w_0(\omega r)$ and $w_0(z)$ is given by (2.10), $g_1(r) = \overline{g_2(r)}$ for $r > 0$ and $\alpha = g_1(0) = \overline{g_2(0)} \neq 0$. We take now $v(z) = g_1(z)z^{\frac{1}{2}+i\nu} - g_2(z)z^{\frac{1}{2}-i\nu}$. Then w_0, v are linearly independent and $\varphi_{1,\omega}(r) = v(r\omega)$ is a solution of (2.2) which satisfies (2.12), by construction and is purely imaginary when ω is real. To prove (2.11) we note that (2.2) has one solution which behaves like $\exp(-\omega r)$ (namely, $\varphi_{0,\omega}$) and one solution which behaves like $\exp(\omega r)$ at ∞ , see [11, Proposition 4] for an elementary proof. Since $\varphi_{1,\omega}$ is independent of $\varphi_{0,\omega}$, then (2.11) holds. \square

Finally we consider the case where $\omega = i\mu$.

Lemma 2.3. *Assume that (1.2) holds. Then for every $\mu > 0$, there exist two solutions $\varphi_{i\mu,0}$ and $\varphi_{i\mu,1}$ of*

$$-\mu^2\varphi(r) - \varphi''(r) + \frac{b}{r^2}\varphi(r) = 0, \quad r \in \mathbb{R}_+ \quad (2.13)$$

satisfying as $r \rightarrow \infty$,

$$e^{-i\mu r}\varphi_{i\mu,0}(r) \rightarrow 1, \quad e^{i\mu r}\varphi_{i\mu,0}(r) \rightarrow i\mu, \quad (2.14)$$

$$e^{i\mu r}\varphi_{i\mu,1}(r) \rightarrow 1, \quad e^{i\mu r}\varphi'_{i\mu,1}(r) \rightarrow -i\mu. \quad (2.15)$$

Proof. It suffices to apply [11, Proposition 5], with $f(x) = -\mu^2$, to (2.13). \square

2.2 The inhomogeneous equation

Lemma 2.4. *Let $\omega \in \mathbb{C}_+$ satisfy $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \pi/2$ and assume that (1.2) holds. Let $\varphi_{\omega,0}$ and $\varphi_{\omega,1}$ be as in Lemmas 2.1, 2.2 and 2.3. Then for $f \in L^2(\mathbb{R}_+)$, every solution of*

$$\omega^2 u(r) - u''(r) + \frac{b}{r^2}u(r) = f(r), \quad r \in \mathbb{R}_+ \quad (2.16)$$

is given by

$$u(r) = c_0\varphi_{\omega,0}(r) + c_1\varphi_{\omega,1}(r) + T_\omega(f) \quad (2.17)$$

where

$$T_\omega(f)(r) = \frac{1}{W(\omega)} \left(\int_0^r \varphi_{\omega,1}(s)f(s) ds \right) \varphi_{\omega,0}(r) + \frac{1}{W(\omega)} \left(\int_r^\infty \varphi_{\omega,0}(s)f(s) ds \right) \varphi_{\omega,1}(r), \quad (2.18)$$

$c_0, c_1 \in \mathbb{C}$ are constants and $W(\omega)$ is the Wronskian of $\varphi_{\omega,0}, \varphi_{\omega,1}$. The map T_ω is a bounded linear operator from $L^2(\mathbb{R}_+)$ to itself and, if ω is real, T_ω is selfadjoint.

Proof. By variation of parameters (2.17) easily follows. Observe that

$$T_\omega f(r) = \int_0^\infty G_\omega(r,s)f(s) ds,$$

where

$$G_\omega(r,s) = \begin{cases} W(\omega)^{-1}\varphi_{\omega,0}(r)\varphi_{\omega,1}(s) & \text{if } s \leq r, \\ W(\omega)^{-1}\varphi_{\omega,0}(s)\varphi_{\omega,1}(r) & \text{if } s \geq r. \end{cases} \quad (2.19)$$

Using Lemmas 2.1, 2.2 and recalling that both solutions are bounded near 0 we obtain $|\varphi_{\omega,0}(r)| \leq Ce^{-(\operatorname{Re}\omega)r}$, $|\varphi_{\omega,1}(r)| \leq Ce^{(\operatorname{Re}\omega)r}$ for every $r > 0$. Therefore

$$|G_\omega(r,s)| \leq C^2 e^{-(\operatorname{Re}\omega)|r-s|}, \quad r > 0, \quad s > 0$$

and the boundedness of T_ω follows. If ω is real, then $\varphi_{\omega,0}, i\varphi_{\omega,1}, iW(\omega)$ are real so that $\overline{G_\omega(r,s)} = G_\omega(s,r)$ and T_ω is selfadjoint. \square

3 Intermediate operators and their spectral properties

Here we characterize all extensions $L_{\min} \subset \tilde{L} \subset L_{\max}$ with non-empty resolvent set and study their spectral properties.

Lemma 3.1. *Let the operator \tilde{L} satisfy $L_{\min} \subset \tilde{L} \subset L_{\max}$. Then $[0, \infty) \subset \sigma(\tilde{L})$.*

Proof. First we prove $(0, \infty) \in \sigma(\tilde{L})$. Let $\eta_n(r)$ be a smooth function equal to 1 in $[n, 2n]$, with support contained in $[n/2, 3n]$ and $0 \leq \eta_n \leq 1$, $|\eta'_n| \leq C/2$, $|\eta''_n| \leq C/n^2$. Given $\varphi_{i\mu,0}$ as in Lemma 2.3 we consider $\psi_n = \eta_n \varphi_{i\mu,0} \in C_0^\infty(\mathbb{R}_+) \subset D(\tilde{L})$. Then $-\mu^2 \psi_n + L\psi_n = -2\eta'_n \varphi'_{i\mu,0} - \eta''_n \varphi_{i\mu,0}$. We have $\|\psi_n\|_2 \approx \sqrt{n}$ and, since $\varphi_{i\mu,0}$ has first and second derivatives bounded near ∞ , $\|(-\mu^2 + L)\psi_n\|_2 \leq Cn^{-1/2}$. Therefore μ^2 is an approximate point spectrum, in other words, $-\mu^2 + L$ cannot have a bounded inverse. Finally, noting that $\sigma(\tilde{L})$ is closed in \mathbb{C} , we have $[0, \infty) \subset \sigma(\tilde{L})$. \square

Lemma 3.2. *Let $L_{\min} \subset \tilde{L} \subset L_{\max}$ and assume that (1.2) and $\rho(\tilde{L}) \neq \emptyset$ hold. Then there exists $c \in \mathbb{C}$ such that defining $(a_1, a_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ by*

$$a_1 = (c + W(\omega)^{-1})\alpha\mu^{i\nu}e^{-\xi\nu} \quad a_2 = (c - W(\omega)^{-1})\bar{\alpha}\mu^{-i\nu}e^{\xi\nu} \quad (3.1)$$

the domain of \tilde{L} is given by

$$D(\tilde{L}) = \left\{ u \in D(L_{\max}) ; \exists C \in \mathbb{C} \text{ s.t. } \lim_{r \downarrow 0} \left| r^{-\frac{1}{2}}u(r) - C(a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0 \right\}. \quad (3.2)$$

Proof. First we show the inclusion “ \subset ” in (3.2). Since, by Lemma 3.1 $[0, \infty) \subset \sigma(\tilde{L})$, we take $\lambda \in \rho(\tilde{L})$ for some $\lambda \in \mathbb{C} \setminus [0, \infty)$. Let $\omega \in \mathbb{C}_+$ satisfy $-\omega^2 = \lambda$. From Lemma 2.4, see (2.17), we have

$$[(\omega^2 + \tilde{L})^{-1}f](r) = c_0(f)\varphi_{\omega,0}(r) + c_1(f)\varphi_{\omega,1}(r) + T_\omega f(r). \quad (3.3)$$

However, $\varphi_{\omega,1} \notin L^2(\mathbb{R}_+)$ and $\varphi_{\omega,0} \in L^2(\mathbb{R}_+)$. Therefore $c_1(f) = 0$ and $c_0(f)$ is a bounded linear functional in $L^2(\mathbb{R}_+)$. Riesz's representation theorem yields $v \in L^2(\mathbb{R}_+)$ such that

$$c_0(f) = \int_0^\infty f(s)v(s)ds. \quad (3.4)$$

If we choose $f = \omega^2 u + Lu$ for $u \in C_0^\infty(\mathbb{R}_+)$, then for r small enough, we see integrating by parts that

$$0 = u(r) = c_0(f)\varphi_{\omega,0}(r) + \frac{1}{W(\omega)} \left(\int_0^\infty \varphi_{\omega,0}(s)f(s)ds \right) \varphi_{\omega,1}(r) = c_0(f)\varphi_{\omega,0}(r).$$

Thus $c_0(f) = 0$ for every $f \in (\omega^2 + L)(C_0^\infty(\mathbb{R}_+))$. This yields that $(\omega^2 + L)v = 0$ and hence

$$v = c\varphi_{\omega,0}, \quad c_0(f) = c \int_0^\infty f(s)\varphi_{\omega,0}(s)ds \quad \text{for some } c \in \mathbb{C}, \quad (3.5)$$

since $v \in L^2(\mathbb{R}_+)$. Consequently, for every $f \in L^2(\mathbb{R}_+)$, $u = (\omega^2 + \tilde{L})^{-1}f$ satisfies

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| u(r) - \left(\int_0^\infty \varphi_{\omega,0}(s)f(s)ds \right) (c\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)) \right| = 0. \quad (3.6)$$

Using (2.4) and (2.12) (with the same notation), we obtain “ \subset ” with $(a_1, a_2) \neq (0, 0)$ given by (3.1) and c given by (3.5).

Conversely, we prove the inclusion “ \supset ” in (3.2). Let $u \in D(L_{\max})$ satisfy

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C' (a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0,$$

where the pair (a_1, a_2) is defined in (3.1) and c in (3.5). By (2.4) and (2.12) we have

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| u(r) - C (c\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)) \right| = 0.$$

Set $\tilde{u} := (\omega^2 + \tilde{L})^{-1}(\omega^2 + L_{\max})u$ and $w := u - \tilde{u}$. Then $(\omega^2 + L)w = 0$ and, since $w \in L^2(\mathbb{R}_+)$, $w = c'\varphi_{\omega,0}$ for some $c' \in \mathbb{C}$. Noting that

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| \tilde{u}(r) - C' (c\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)) \right| = 0,$$

we obtain

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| c'\varphi_{\omega,0}(r) - (C - C') (c\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)) \right| = 0$$

or

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| (c' - c(C - C')) \varphi_{\omega,0}(r) - (C - C')W(\omega)^{-1}\varphi_{\omega,1}(r) \right| = 0.$$

By (2.4) and (2.12) again we deduce that $c' = 0$, hence $u = \tilde{u} \in D(\tilde{L})$. □

In view of Lemma 3.2, we define intermediate operators between L_{\min} and L_{\max} as follows.

Definition 1. Let $A := (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then

$$\begin{cases} D(L_A) := \left\{ u \in D(L_{\max}) ; \exists C \in \mathbb{C} \text{ s.t. } \lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C (a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0 \right\}, \\ L_A u = Lu. \end{cases}$$

Remark 3.1. If \tilde{L} satisfies $L_{\min} \subset \tilde{L} \subset L_{\max}$ and $\rho(\tilde{L}) \neq \emptyset$, by Lemma 3.2 there exists a pair $A = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that \tilde{L} coincides with L_A . Moreover, if $a'_1 = ca_1$ and $a'_2 = ca_2$ for some $c \in \mathbb{C} \setminus \{0\}$, then $L_A = L_{A'}$. This implies that the map

$$A \in \mathbb{CP}_1 \mapsto L_A \in \{\tilde{L} ; L_{\min} \subset \tilde{L} \subset L_{\max} \text{ \& } \rho(\tilde{L}) \neq \emptyset\}$$

is well-defined and one to one, where \mathbb{CP}_1 denotes the Riemann sphere (or the one-dimensional complex projective space). Note that it is known in a field of mathematical physics that there exists a bijective map

$$\mathbb{RP}_1(\cong S^1) \rightarrow \{\tilde{L} ; L_{\min} \subset \tilde{L} \subset L_{\max} \text{ \& } \tilde{L} \text{ is selfadjoint}\}.$$

See Proposition 3.5 for more explanation.

In order to compute the spectrum of L_A we need the following preliminary result.

Lemma 3.3. *Let $\omega = \mu e^{i\xi} \in \mathbb{C}_+$, $|\xi| < \pi/2$. Then $(\omega^2 + L_A)$ is invertible if and only if $\varphi_{\omega,0} \notin D(L_A)$.*

Proof. Let us assume that $\varphi_{\omega,0} \notin D(L_A)$ so that $\omega^2 + L_A$ is injective. By (2.4) this is equivalent to saying that

$$\begin{vmatrix} \alpha\mu^{i\nu}e^{-\xi\nu} & \overline{\alpha}\mu^{-i\nu}e^{\xi\nu} \\ a_1 & a_2 \end{vmatrix} \neq 0 \quad (3.7)$$

Let $f \in L^2(\mathbb{R}_+)$ and $u = c_0(f) + T_\omega f$, where $c_0(f)$ is defined in (3.5). Then (3.6) holds $u \in D(L_B)$ where $B = (b_1, b_2)$ and

$$b_1 = (c + W(\omega)^{-1})\alpha\mu^{i\nu}e^{-\xi\nu} \quad b_2 = (c - W(\omega)^{-1})\overline{\alpha}\mu^{i\nu}e^{\xi\nu}.$$

The system $b_1 = \kappa a_1, b_2 = \kappa a_2$ has a unique solution (c, κ) because of (3.7). With this choice, $u \in D(L_B) = D(L_A)$ and $(\omega^2 + L_A)^{-1}f = c_0(f) + T_\omega f$ is bounded because of (3.5) and Lemma 2.4. \square

To formulate the main theorem of this paper we introduce the set

$$\begin{aligned} S(\kappa) &= \left\{ -\rho e^{i\theta} \in \mathbb{C} : \rho^{-i\nu} e^{\theta\nu} = \kappa e^{2i\eta} \right\} \\ &= \left\{ -\rho_j e^{i\theta} \in \mathbb{C} : \theta = \frac{\log|\kappa|}{\nu}, \rho_j = e^{\frac{\eta+2j\pi}{\nu}}, j \in \mathbb{Z} \right\}, \end{aligned} \quad (3.8)$$

where $\kappa \in \mathbb{C} \setminus \{0\}$ and $\alpha = |\alpha|e^{i\eta}$ is defined in (2.1). Note that $S(\kappa)$ consists of double sequence $\{(z_j), j \in \mathbb{Z}\}$ lying on the half line $\{z = -\rho e^{i\theta}\}$, such that $|z_j| \rightarrow \infty$ as $j \rightarrow +\infty$ and $|z_j| \rightarrow 0$ as $j \rightarrow -\infty$. The above angle θ is independent of α and the moduli of the points z_j depend only on ν and $\eta = \arg(\alpha)$. From (2.1) we see that $\eta \rightarrow \pi/2$ as $\nu \rightarrow 0$ and, using [1, 6.1.44, p.257],

$$\eta = -\nu \log \nu + (1 - \log 2)\nu + \pi/4 + o(1)$$

as $\nu \rightarrow +\infty$.

Theorem 3.4. *The following assertions hold*

(i) *Assume $a_1 \neq 0, a_2 \neq 0$ and let $\kappa = \frac{a_1}{a_2}$. If*

$$|\kappa| \in (e^{-\nu\pi}, e^{\nu\pi}), \quad (3.9)$$

then

$$\sigma(L_A) = [0, \infty) \cup S(\kappa).$$

Moreover, $S(\kappa)$ coincides with the set of all eigenvalues of L_A .

(ii) *If A does not satisfy condition in (i), then*

$$\sigma(L_A) = [0, \infty).$$

Proof. Lemma 3.1 yields $[0, \infty) \subset \sigma(L_A)$. If $\omega = \mu e^{i\xi} \in \mathbb{C}_+$, $|\xi| < \pi/2$, Lemma 3.3 says that $\lambda = -\omega^2 \in \sigma(L_A)$ if and only if $\varphi_{\omega,0} \in D(L_A)$. By (3.7) this happens if and only if

$$a_1 \overline{\alpha} = a_2 \alpha \mu^{2i\nu} e^{-2\xi\nu} \quad (3.10)$$

or $\lambda \in S(\kappa, \alpha)$. Since $|2\xi| < \pi$ this equation can be satisfied only when (3.9) holds. Finally, the assertion concerning the eigenvalues follow from Lemmas 2.3, 3.3. \square

Finally, we characterize the adjoint of L_A .

Proposition 3.5. *Let $A = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then $(L_A)^* = L_B$ where $B = (b_1, b_2)$ and $b_1 = \bar{a}_2$, $b_2 = \bar{a}_1$. L_A is selfadjoint if and only if $|a_1| = |a_2|$.*

Proof. Theorem 3.4 yields the existence of $\omega > 0$ such that $\omega^2 + L_A$ is invertible. From Lemma 3.2 we know that

$$(\omega^2 + L_A)^{-1}f = c \left(\int_0^\infty \varphi_{\omega,0}(s)f(s) ds \right) \varphi_{\omega,0} + T_\omega f$$

for a suitable $c \in \mathbb{C}$ and then (3.1) with $\mu = \omega$ and $\xi = 0$ yields

$$a_1 = (c + W(\omega)^{-1})\alpha\omega^{i\nu} \quad a_2 = (c - W(\omega)^{-1})\bar{\alpha}\omega^{-i\nu}.$$

Since, by Lemma 2.4, T_ω is selfadjoint we obtain

$$(\omega^2 + (L_A)^*)^{-1}f = \bar{c} \left(\int_0^\infty \varphi_{\omega,0}(s)f(s) ds \right) \varphi_{\omega,0} + T_\omega f$$

and therefore $(L_A)^* = L_B$ where

$$b_1 = (\bar{c} + W(\omega)^{-1})\alpha\omega^{i\nu} = \bar{a}_2 \quad b_2 = (\bar{c} - W(\omega)^{-1})\bar{\alpha}\omega^{-i\nu} = \bar{a}_1$$

since $W(\omega)$ is purely imaginary. Finally, L_A is selfadjoint if and only if $\bar{a}_2 = ca_1$, $\bar{a}_1 = ca_2$ for a suitable $c \in \mathbb{C} \setminus \{0\}$ and this happens if and only if $|a_1| = |a_2|$. \square

Remark 3.2. Four cases appear in the description of $\sigma(L_A)$.

Case I. Assume that L_A is selfadjoint. By Proposition 3.5, we have $|\kappa| = 1$ and $\theta = 0$. It follows from Theorem 3.4 that every selfadjoint extension of L_{\min} has infinitely many eigenvalues and its spectrum is unbounded both from above and below, see Figure 1.

Case II. Next we consider the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}} \right].$$

that is, $\theta \in [-\pi/2, \pi/2]$. In this case, $\rho(-L_A)$ does not contain $\overline{\mathbb{C}_+} \setminus \{0\}$, see Figure 2. Therefore, $-L_A$ does not generate an analytic semigroup on $L^2(\mathbb{R}_+)$.

Case III. In the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in (e^{-\nu\pi}, e^{\nu\pi}) \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}} \right],$$

we have $\theta \in (-\pi, \pi) \setminus [-\pi/2, \pi/2]$ (see Figure 3). Hence one can expect that $-L_A$ generates an analytic semigroup on $L^2(\mathbb{R}_+)$. Indeed, we prove in Proposition 4.1 that $-L_A$ generates a bounded analytic semigroup of angle $\pi/2 - |\theta|$.

Case IV. Finally we consider the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in [0, \infty] \setminus (e^{-\nu\pi}, e^{\nu\pi}).$$

Here we use $|\kappa| = \infty$ if $a_1 = 0$ and $|\kappa| = 0$ if $a_2 = 0$. By Theorem 3.4 (ii) we have $\sigma(L_A) = [0, \infty)$, see Figure 4. As in Case III, we prove that $-L_A$ generates a bounded analytic semigroup on $L^2(\mathbb{R}_+)$ of angle $\pi/2$.

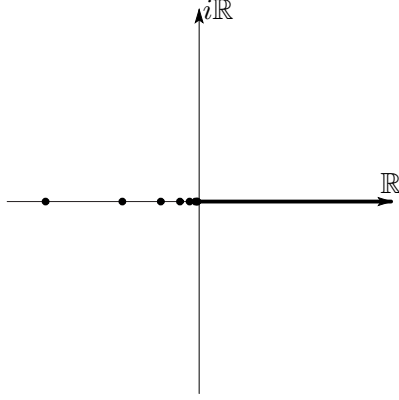


Figure 1 : Selfadjoint case $\theta = 0$ (Case I)

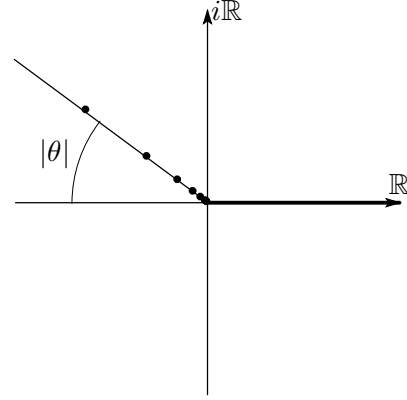


Figure 2 : $|\theta| \leq \pi/2$ (Case II)

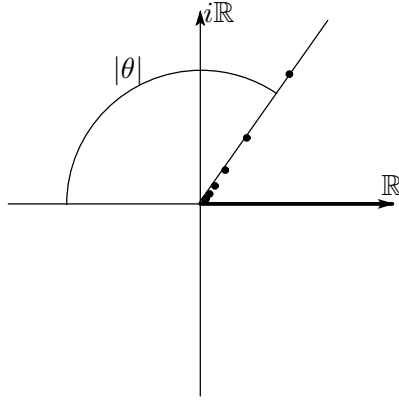


Figure 3 : $\pi/2 < |\theta| < \pi$ (Case III)

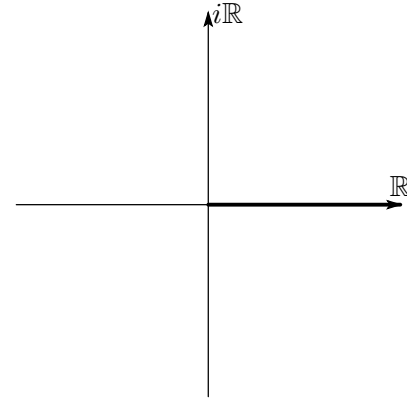


Figure 4 : $|\theta| \geq \pi$ (Case IV)

4 Generation of analytic semigroups

We characterize when L_A generates an analytic semigroup.

Theorem 4.1. *Let L_A be defined in Definition 1. Then $-L_A$ generates a bounded analytic semigroup $\{T_A(z)\}$ on $L^2(\mathbb{R}_+)$ if and only if a_1 and a_2 satisfy*

$$|\kappa| = \frac{|a_2|}{|a_1|} \in [0, \infty] \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}} \right]. \quad (4.1)$$

Moreover, if $\theta = \frac{\log |\kappa|}{\nu}$, the maximal angle of analyticity θ_A of $\{T_A(z)\}$ is given by

$$\theta_A := \begin{cases} |\theta| - \frac{\pi}{2} & \text{if } |\kappa| \in (e^{-\nu\pi}, e^{\nu\pi}) \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}} \right], \\ \frac{\pi}{2} & \text{otherwise.} \end{cases}$$

Setting

$$\Sigma(\theta) := \{z \in \mathbb{C} \setminus \{0\} ; |\operatorname{Arg} z| < |\theta|\}.$$

from Theorem 3.4, we immediately obtain

Lemma 4.2. $\Sigma(\pi/2 + \theta_A) \subset \rho(-L_A)$. In particular, $\overline{\mathbb{C}}_+ \setminus \{0\} \subset \rho(-L_A)$ if and only if a_1 and a_2 satisfy (4.1).

To prove Theorem (4.1), we use a scaling argument. It worth noticing that if $a_1 \neq 0$ and $a_2 \neq 0$, then $D(L_A)$ is not invariant under scaling $u(r) \mapsto u(s_0 r)$ for some $s_0 > 0$ in spite of the scale invariant property of $D(L_{\min})$ and $D(L_{\max})$. This means that the scale symmetry of L_A (with $s \in (0, \infty)$) is broken. However, there exists a subgroup G of $(0, \infty)$ such that the scale symmetry of L_A with $s \in G$ is still true.

Lemma 4.3. For $\nu > 0$, we define

$$G(\nu) := \left\{ e^{\frac{m\pi}{\nu}} ; m \in \mathbb{Z} \right\}. \quad (4.2)$$

Assume that $a_1 \neq 0$ and $a_2 \neq 0$. Then $D(L_A)$ is invariant under the scaling $u(r) \mapsto u(sr)$ if and only if $s \in G(\nu)$. On the other hand, if $a_1 = 0$ or $a_2 = 0$, then $D(L_A)$ is invariant under the scaling $u(r) \mapsto u(sr)$ for every $s \in (0, \infty)$.

Proof. Fix $A = (a_1, a_2)$ with $a_1 \neq 0$ and $a_2 \neq 0$ and let $u \in D(L_A)$ satisfy

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C (a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0$$

for some $C \neq 0$. Then $u(sr) \in D(L_A)$ if and only if

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(sr) - C' (a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0$$

for some C' . This is equivalent to saying that

$$\lim_{r \downarrow 0} \left| C (a_1 (sr)^{i\nu} + a_2 (sr)^{-i\nu}) - C' (a_1 r^{i\nu} + a_2 r^{-i\nu}) \right| = 0, \quad (4.3)$$

or

$$Cs^{i\nu} = C' = Cs^{-i\nu}. \quad (4.4)$$

We obtain $\log s \in (\pi/\nu)\mathbb{Z}$, or equivalently, $s \in G(\nu)$. The cases $a_1 = 0$ or $a_2 = 0$ are similar. \square

Proof of Theorem 4.1. Assume (4.1). For $0 < \varepsilon < \theta_A$ let

$$\Sigma_\varepsilon := \left\{ \lambda \in \overline{\Sigma(\pi/2 + \theta_A - \varepsilon)} ; 1 \leq |\lambda| \leq e^{\frac{2\pi}{\nu}} \right\} \subset \rho(-L_A).$$

Since Σ_ε is compact in \mathbb{C} , $\|(\lambda + L_A)^{-1}\|$ is bounded in Σ_ε . Therefore we have

$$\|(\lambda + L_A)^{-1}\| \leq \frac{M_\varepsilon}{|\lambda|}, \quad \lambda \in \Sigma_\varepsilon.$$

Observe that by Lemma 4.3 the dilation operator $(I_s u)(x) := s^{\frac{1}{2}} u(sx)$ satisfies $\|I_s u\|_{L^2(\mathbb{R}_+)} = \|u\|_{L^2(\mathbb{R}_+)}$ and

$$L_A I_s = s^2 I_s L_A, \quad s \in G(\nu). \quad (4.5)$$

Let $\lambda \in \Sigma(\pi/2 + \theta_A - \varepsilon)$. Taking $s \in G(\nu)$ as

$$\log s_0 \in \left[-\frac{\log |\lambda|}{2}, \frac{\pi}{\nu} - \frac{\log |\lambda|}{2} \right) \cap \frac{\pi}{\nu} \mathbb{Z} \neq \emptyset, \quad (4.6)$$

we see that $s_0^2 \lambda \in \Sigma_\varepsilon$, and hence, we have

$$\|(s_0^2 \lambda + L_A)^{-1}\| \leq \frac{M_\varepsilon}{|s_0^2 \lambda|}.$$

Using (4.5) with (4.6), we obtain

$$\|(\lambda + L_A)^{-1}\| = \|(\lambda + s_0^{-2} I_{s_0^{-1}} L_A I_{s_0})^{-1}\| = s_0^2 \|I_{s_0^{-1}} (s_0^2 \lambda + L_A)^{-1} I_{s_0}\| \leq \frac{s_0^2 M_\varepsilon}{|s_0^2 \lambda|} = \frac{M_\varepsilon}{|\lambda|}.$$

Therefore $-L_A$ generates a bounded analytic semigroup on $L^2(\mathbb{R}_+)$ of angle θ_A . The optimality of θ_A follows from Lemma 4.2.

On the other hand, if (4.1) is violated, then Proposition 4.2 implies that $-L_A$ does not generate an analytic semigroup on $L^2(\mathbb{R}_+)$. \square

Remark 4.1. In the case $|\kappa| = e^{\frac{\nu\pi}{2}}$ or $|\kappa| = e^{-\frac{\nu\pi}{2}}$, we do not know whether the operator $-L_A$ generates a C_0 -semigroup on $L^2(\mathbb{R}_+)$. We point out that if $-L_A$ generates a C_0 -semigroup, then it cannot be (quasi) contractive because Hardy's inequality does not hold on $C_0^\infty(\mathbb{R}_+)$, since $b < -\frac{1}{4}$.

5 Remarks on the N -dimensional case

Here we consider the N -dimensional Schrödinger operators, $N \geq 2$,

$$L = -\Delta + \frac{b}{|x|^2}, \quad b \in \mathbb{R}$$

As in one dimension we define

$$\begin{aligned} D(L_{\min}) &:= C_0^\infty(\mathbb{R}^N \setminus \{0\}), \\ D(L_{\max}) &:= \{u \in L^2(\mathbb{R}^N) \cap H_{\text{loc}}^2(\mathbb{R}^N \setminus \{0\}) ; Lu \in L^2(\mathbb{R}^N)\}. \end{aligned}$$

Hardy's inequality

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \quad (5.1)$$

implies the existence of a nonnegative selfadjoint extension of L_{\min} , namely the Friedrichs extension, for $b \geq -(\frac{N-2}{2})^2$. Therefore in this section we assume

$$b < -\left(\frac{N-2}{2}\right)^2. \quad (5.2)$$

Using Proposition 4.1 we obtain the following result.

Proposition 5.1. *Assume that (5.2) holds. Then there exist infinitely many intermediate operators between L_{\min} and L_{\max} which are negative generators of analytic semigroups on $L^2(\mathbb{R}^N)$.*

To prove Proposition 5.1 we expand $f \in L^2(\mathbb{R}^N)$ in spherical harmonics

$$f = \sum_{j=0}^{\infty} F_j(G_j f).$$

where $F_j : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}^N)$ and $G_j : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}_+)$ are defined by

$$\begin{aligned} F_j g(x) &:= |x|^{-\frac{N-1}{2}} g(|x|) Q_j(\omega), \quad g \in L^2(\mathbb{R}_+), \\ G_j f(r) &:= r^{\frac{N-1}{2}} \int_{S^{N-1}} f(r, \omega) Q_j(\omega) d\omega, \quad f \in L^2(\mathbb{R}^N). \end{aligned}$$

Here $\{Q_j ; j \in \mathbb{N}\}$ is a orthonormal basis of $L^2(S^{N-1})$ consisting of spherical harmonics Q_j of order n_j . Q_j is an eigenfunction of Laplace-Beltrami operator $\Delta_{S^{N-1}}$ with respect to the eigenvalue $-\lambda_j = -n_j(N-2+n_j)$, see e.g., [15, 30, Chapter IX] and also [13, Ch. 4, Lemma 2.18].

Lemma 5.2. *For every $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the following assertions hold*

(i) $\|F_j g\|_{L^2(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}_+)}$ for every $g \in L^2(\mathbb{R}_+)$, $\|G_j f\|_{L^2(\mathbb{R}_+)} \leq \|f\|_{L^2(\mathbb{R}^N)}$ for every $f \in L^2(\mathbb{R}^N)$;

(ii) $G_j F_j = 1_{L^2(\mathbb{R}_+)}$ and $F_j G_j[D(L_{\min})] \subset D(L_{\min})$;

(iii) for every $v \in C_0^\infty(\mathbb{R}_+)$,

$$G_j[L(F_j v)](r) = -v''(r) + \frac{b_j}{r^2} v(r),$$

where

$$b_j := b + \left(\frac{N-2}{2}\right)^2 - \frac{1}{4} + \lambda_{n_j}.$$

Proof. (i) and (ii) follow easily by direct computation. We only prove (iii). Let $v \in C_c^\infty(\mathbb{R}_+)$. Observing that

$$L = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{|x|} \frac{\partial}{\partial r} + \frac{b}{|x|^2} - \frac{1}{|x|^2} \Delta_{S^{N-1}},$$

we deduce

$$\begin{aligned} L(F_j v)(x) &= L\left(|x|^{-\frac{N-1}{2}} v(|x|) Q(\omega)\right) \\ &= |x|^{-\frac{N-1}{2}} \left[-v''(|x|) + \left(b + \frac{(N-1)(N-3)}{4} + \lambda_{n_j}\right) \frac{1}{|x|^2} v(|x|)\right] Q(\omega). \end{aligned}$$

Therefore,

$$G_j[L(F_j v)](r) = -v''(r) + \frac{b_j}{r^2} v(r).$$

□

Proof of Proposition 5.1. If $j \in \mathbb{N}_0$ satisfies $b_j \geq \frac{1}{4}$, then from Lemma 5.2 (ii) $L_{j,\min} := F_j L_{\min} G_j$ is nonnegative, and hence there exists a Friedrichs extension $L_{j,F}$ of $L_{j,\min}$. This implies that

$$\|(\lambda - L_{j,F})^{-1}\| \leq \frac{1}{|\lambda|} \quad \lambda \in \mathbb{C}_+.$$

If $j \in \mathbb{N}$ satisfies $b_j < -\frac{1}{4}$, then we choose $A_j = (a_{1,j}, a_{2,j}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ satisfying (4.1) with $\nu_j = \sqrt{-b_j - 1/4}$. By Proposition 4.1, we have

$$\|(\lambda - L_{j,A_j})^{-1}\| \leq \frac{M_j}{|\lambda|} \quad \lambda \in \mathbb{C}_+.$$

Now we define the operator \tilde{L} between L_{\min} and L_{\max} as follows:

$$D(\tilde{L}) := \left(\bigoplus_{b_j \geq -1/4} F_j D(L_{j,F}) \right) \oplus \left(\bigoplus_{b_j < -1/4} F_j D(L_{j,A_j}) \right);$$

note that $\tilde{L} \supset L_{\min}$ is verified by Lemma 5.2 (ii). Then we see $\lambda - \tilde{L}$ is injective for every $\lambda \in \mathbb{C}_+$. In fact, if $\lambda - \tilde{L}u = 0$ for $u \in D(\tilde{L})$, then for every $j \in \mathbb{N}$, by the definition of $D(\tilde{L})$ it follows from Lemma 5.2 (iii) that $(\lambda - L_{j,F})u_j = 0$ with $u_j := G_j u \in D(L_{j,F})$ when $b_j \geq -1/4$ and $(\lambda - L_{j,A_j})u_j = 0$ with $u_j := G_j u \in D(L_{j,A_j})$ when $b_j < -1/4$. This implies that $u_j = 0$ for every $j \in \mathbb{N}$, hence $u = \sum_{j \in \mathbb{N}} F_j u_j = 0$.

Moreover, for every $f \in L^2(\mathbb{R}^N)$, we have $f = \lambda u - \tilde{L}u$, where we set

$$u := \sum_{b_j \geq -1/4} F_j (\lambda - L_{j,F})^{-1} G_j f + \sum_{b_j < -1/4} F_j (\lambda - L_{j,A_j})^{-1} G_j f \in D(\tilde{L}).$$

Since the $\{j \in \mathbb{N} ; b_j < -1/4\}$ is finite, $\tilde{M} := \max\{M_j ; b_j < -1/4\}$ is also finite. Hence it follows from Lemma 5.2 (i) that for every $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} \|u\|_{L^2(\mathbb{R}^N)}^2 &= \sum_{b_j \geq -1/4} \|F_j (\lambda - L_{j,F})^{-1} G_j f\|_{L^2(\mathbb{R}^N)}^2 + \sum_{b_j < -1/4} \|F_j (\lambda - L_{j,A_j})^{-1} G_j f\|_{L^2(\mathbb{R}^N)}^2 \\ &= \sum_{b_j \geq -1/4} \|(\lambda - L_{j,F})^{-1} G_j f\|_{L^2(\mathbb{R}_+)}^2 + \sum_{b_j < -1/4} \|(\lambda - L_{j,A_j})^{-1} G_j f\|_{L^2(\mathbb{R}_+)}^2 \\ &\leq \sum_{b_j \geq -1/4} \frac{1}{|\lambda|} \|G_j f\|_{L^2(\mathbb{R}_+)}^2 + \sum_{b_j < -1/4} \frac{M_j}{|\lambda|} \|G_j f\|_{L^2(\mathbb{R}_+)}^2 \\ &\leq \frac{\tilde{M}}{|\lambda|} \left(\sum_{b_j \geq -1/4} \|F_j G_j f\|_{L^2(\mathbb{R}^N)}^2 + \sum_{b_j < -1/4} \|F_j G_j f\|_{L^2(\mathbb{R}^N)}^2 \right) \\ &= \frac{\tilde{M}}{|\lambda|} \|f\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Therefore \tilde{L} is closed, $\mathbb{C}_+ \subset \rho(-\tilde{L})$ and

$$\|(\lambda - \tilde{L})^{-1}\| \leq \frac{\tilde{M}}{|\lambda|}.$$

This implies that $-\tilde{L}$ generates a bounded analytic semigroup on $L^2(\mathbb{R}^N)$. Since we can choose all of A_j satisfying (4.1), we can produce infinitely many (negative) generators between L_{\min} and L_{\max} . \square

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